Intertime jump statistics of state-dependent Poisson processes

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A method to obtain the probability distribution of the interarrival times of jump occurrences in systems driven by state-dependent Poisson noise is proposed. Such a method uses the survivor function obtained by a modified version of the master equation associated to the stochastic process under analysis. A model for the timing of human activities shows the capability of state-dependent Poisson noise to generate power-law distributions. The application of the method to a model for neuron dynamics and to a hydrological model accounting for land-atmosphere interaction elucidates the origin of characteristic recurrence intervals and possible persistence in state-dependent Poisson models.

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I. INTRODUCTION

Continuous time processes with jump transitions are very common in different fields, such as queuing theory and storage problems with application to insurance risk and stock market modeling [1–3]. These types of stochastic processes have also been extensively used in theoretical models of neuron dynamics, where the voltage across nerve membrane is assumed to follow a trajectory with sudden drops caused by nerve excitations [4–6], as well as in hydrology to describe infiltration events in the soil water balance [7–13]. Further uses can be found in the study of noise-driven transport [14], population dynamics [15], extreme event dynamics [16], occurrence of fires in ecosystems [17,18], and in the general theory of stochastic processes [1,19–21].

One of the most important features of jump processes is the recurrence time of jump occurrence, i.e., the distribution of the times τ between successive jumps, $p_\tau(\tau)$. Although in many models jumps are often assumed to have constant probability of occurrence (i.e., Poisson process), so that $p_\tau(\tau)$ is simply exponential [1–3,7,8], in many systems the probability of jumps depends on the actual state of the system. In such cases the frequency of jump occurrence becomes state dependent [9,18,22], and the probability distribution (pdf) of the times between two successive jumps is no longer exponential. The state dependence of the Poisson process can be the cause of power-law tails of jump intertime distributions, a fact that could be important in modeling many phenomena, such as human action executions [23–26], large-earthquakes recurrence [27,28], and solar flaring rates [29,30].

The purpose of the present work is to develop a method to obtain $p_\tau(\tau)$ for general nonlinear processes driven by deterministic laws and intermittently forced by state-dependent marked Poisson process. After developing the general theoretical framework (Sec. II), we present three applications related to the timing of human activities (Sec. III), neuron dynamics (Sec. IV), and a hydrological model for rainfall persistence (Sec. V).

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II. MATHEMATICAL THEORY

We deal with dynamical systems that can be described in terms of a single representative stochastic variable, $x(t)$, which follows a deterministic trajectory perturbed by jumps of random timing and amplitudes. The jumps are modeled as a state-dependent compound Poisson process $F(x,t)=\sum_{i=1}^{N(t)} y_i \delta(t-t_i)$, where $\delta(\cdot)$ is the Dirac delta function [9]. The times $\{t_i\}$ $(i=1,2,\ldots)$ denote a random variable that expresses the arrival time of the ith event of a state-dependent Poisson counting process $N(t)=\Pi(\int_0^t \lambda[x(s)]ds)$, where $\Pi(t)$ is a unit-rate Poisson process and $\lambda=\lambda[x(t)]$ is the jump occurrence rate. The amplitudes of the jumps, y_i, are mutually independent random variables with probability distribution function $h(y;x)$ that in general can be state dependent as well. In other words, the transition probability density per unit time $W(z|x)$ for a flipping from the state x into the state z takes the form $W(z|x)=h(z-x;x)\lambda(x)$ and it is normalized over z to unity [31].

According to the previous modeling assumptions, the state of the system, $x(t)$, is described by the stochastic equation

$$\frac{dx}{dt} = f(x) + F(x,t), \tag{1}$$

where $f(x)$ is a deterministic function.

The probability distribution function of $x(t)$, $p(x,t)$, satisfies the differential Chapman-Kolmogorov forward equation [1,12,13]

$$\frac{\partial}{\partial t}p(x,t) = -\frac{\partial}{\partial x}(f(x)p(x,t)) - \lambda(x)p(x,t) + \int_{-\infty}^{+\infty} \lambda(x-z)p(x-z,t)h(z;x-z)dz, \tag{2}$$

where the terms on the r.h.s. are the contributions due to the drift $f(x)$, the jump occurrences which cause the process to leave the current trajectory, and the jumps that bring the system to the state x, respectively.

The process defined by Eq. (1) can be also interpreted as a composition of sequences of deterministic trajectories of different durations. Each trajectory starts from a random value x_s and ends at a random value x_e. When the system reaches steady state, the ending extremes (x_e) of the trajectories result to be distributed as follows:

$$p_e(x) = \lambda(x)p(x)/\langle\lambda\rangle, \tag{3}$$

where $\langle\lambda\rangle$ is the average of the frequency of jump occurrences and $p(x)$ is the steady state pdf of Eq. (2). Such a distribution can be obtained by considering the probability of being in $x+dx$ and (independently) jumping during a time interval dt from that level, i.e., $p(x)dx\lambda(x)dt$, and then normalizing it by the total probability of jumping during the interval dt, $\int_{-\infty}^{+\infty}\lambda(x)p(x)dxdt$. The distribution of the starting points (x_s) of the trajectories can be found by convolving $p_e(x)$ with the jump distribution, i.e.,

$$p_s(x) = \int_{-\infty}^{+\infty} p_e(x-z)h(z;x-z)dz. \tag{4}$$

The duration of the trajectories between x_s and x_e is distributed as $p_\tau(\tau)$ and is the main focus of the study.

In order to determine $p_\tau(\tau)$ it is useful to introduce a modified version of Eq. (2),

$$\frac{\partial}{\partial t}p_\lambda(x,t) = -\frac{\partial}{\partial x}[f(x)p_\lambda(x,t)] - \lambda(x)p_\lambda(x,t). \tag{5}$$

The absence of the integral term in Eq. (5) suggests that $p_\lambda(x,t)$ is associated to a process similar to that of Eq. (1), with the difference that trajectories stop after the first jump. Therefore, when Eq. (5) is solved with the initial condition $p_\lambda(x,0)=p_s(x)$, the related solution, $p_\lambda(x,t)$, represents the pdf of x at time t of the ensemble of trajectories originated from x_s [with distribution $p_s(x)$] that have evolved deterministically as $\dot{x}=f(x)$ before the occurrence of a jump. The occurrence of jumps in the ensemble of trajectories starting from x_s causes $p_\lambda(x,t)$ to lose mass at a rate $\int_{-\infty}^{+\infty}\lambda(x)p_\lambda(x,t)dx$. Therefore the area $\int_{-\infty}^{+\infty}p_\lambda(x,t)dx$ tends to zero as t goes to $+\infty$, giving the fraction of trajectories lasting (without a jump) up to time t. Consequently, $p_\lambda(x,\tau)$ is related to the probability, $\mathcal{F}(\tau)$, that the process $x(t)$ has not experienced a jump up to time $t=\tau$ as follows:

$$\mathcal{F}(\tau) = \int_{-\infty}^{+\infty} p_\lambda(x,\tau)dx, \tag{6}$$

which can be interpreted as a *survivor function* [22,32,33]. The pdf of the times between jumps, $p_\tau(\tau)$, is related to $\mathcal{F}(\tau)$ as follows:

$$p_\tau(\tau) = -\frac{d}{d\tau}\mathcal{F}(\tau). \tag{7}$$

It is worth recalling that the above method is valid for systems in steady-state, for which it is possible to define the pdf $p_s(x)$ from Eq. (4).

We underline that the previous derivations generalize the theory developed in [22]. In fact the systems studied in [22] are forced by jumps that reinitialize $x(t)$ always at the same value, so that the jumps occur as a renewal process, since, after the first jump, the intertimes between consecutive jumps are independent and identically distributed. Conversely, here we deal with processes in which the jumps in general bring the system to different random states, generating a dependence of the times between successive jumps on the previous history of the process.

In the following we will discuss three examples to present applications of the introduced method and to show the effects of the state-dependent Poisson noise on the system dynamics and their jump intertimes. We note that the first application (Sec. III) regards a system that could be studied using the method developed in Ref. [22]. Nonetheless, given its different application and its paradigmatic ability to generate power-law tails in the intertime pdf, we briefly present it here.

III. A POSSIBLE MODEL FOR HUMAN BEHAVIOR

A lot of attention has been recently devoted to modeling the timing of human activities (e.g., emails, phone calls, investments) and their impact in social, technological, and economical dynamics [23–26]. Recent analyses have shown that the timing of human activities does not tend to follow a purely random pattern as predicted by the classic Poisson process, but is affected by feedbacks causing delays and accelerations in relation to their importance, resulting in possible clustering and heavy tails in the distribution of the intertimes [23–26]. Here we comment on the fact that such patterns can be still interpreted as a Poisson process, as long as its rate is related to a suitable state variable, $x(t)$.

Consider email use: it is reasonable to assume that, upon the reading of emails, our mind becomes fully concentrated on it and therefore the "attention," $x(t)$, to the other many tasks is set to zero. The probability of performing an action related to the email use, such as replying to emails, is at a maximum at that moment, while it progressively decreases as the attention to the other tasks grows back again. As a result, each time our mind is called to focus on email, the attention to other tasks, $x(t)$, behaves as a state-dependent renewal process with negative ageing [1]. For simplicity, we assume that $x(t)$ is set to zero by the actual use of the emails and increases linearly in time at a constant rate, i.e., $f(x)=k=\text{const}$ $(0<k<1)$. The rate of email use is taken as inversely proportional to $x(t)$, as $\lambda(x)=1/(1+x)$, since when the attention to other tasks is higher the probability to use the email decreases. Once the email is used, $x(t)$ is reset to zero and the process is assumed to continue like this indefinitely.

As seen in Fig. 1, the system is characterized by bursts of frequently occurring jumps separated by long periods without jumps. This generates power-law tails in the pdf of the time of email use [23,26]. It is easy to show that in the previous specific case $p_\tau(\tau)$ is the generalized Pareto distribution, $p_\tau(\tau)=(1+k\tau)^{-1-1/k}$ [22].

The previous behavior can be generalized considering a stationary process in which the attention level $x(t)$ is always positive and with jumps that reinitialize the system at $x=0$ [i.e., $h(y,x)=\delta(y-x)$] [22]. In such a case, the point process of the jump occurrences becomes a state-dependent Poisson

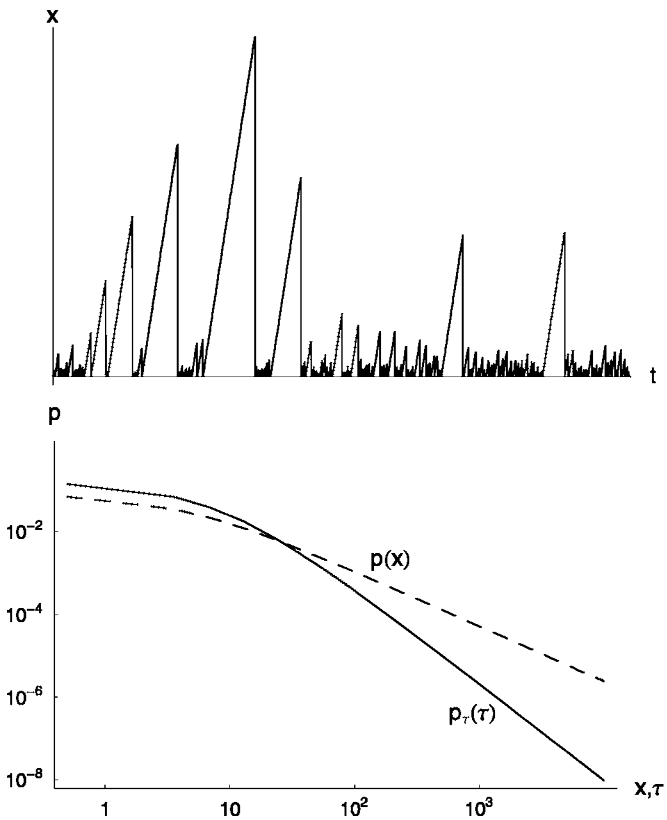

FIG. 1. Example of time series of the variable $x(t)$ (top) and correspondent steady-state probability distributions of x, $p(x)$, and of intertimes between consecutive actions, $p_\tau(\tau)$, in a log-log plot (bottom). The presence of long periods between actions among frequent executions is evident and is the cause of the power law in $p_\tau(\tau)$. Parameters: $k=1/2$, $a=1/6$, $b=1$, and $c=1/4$.

process and the distribution of the times between jumps reads

$$p_\tau(\tau) = \lambda[x(\tau)]\exp\left\{-\int_0^\tau \lambda[x(u)]du\right\}, \tag{8}$$

where $x(\tau)$ can be derived inverting the relation $\tau=\int_0^x du/f(u)$. If $f(x)$ and $\lambda(x)$ have a general form such that the resulting frequency of jumps in terms of τ is $\lambda(\tau)=g(\tau)/(a+b\tau)$, $g(\tau)$ being an arbitrary function and a and b positive parameters, the jump intertime pdf becomes

$$p_\tau(\tau) = Cg(\tau)(a+b\tau)^{-(1+g(\tau)/b)}\exp\left\{-\int_0^\tau \ln[(a+bu)^{1/b}] \times\left[\frac{d}{du}g(u)\right]du\right\}, \tag{9}$$

where C is a normalization constant. Thus, $p_\tau(\tau)$ is a stretched exponential distribution [34], that becomes a Pareto when $g(\tau)$ is constant.

This simple analysis shows that the scaling behavior of the intertime occurrence of particular phenomena (e.g., human action execution, large earthquake recurrence, solar flaring rates, fires in ecosystems) might be interpreted as the result of the dependence of their rate of occurrence on a variable, $x(t)$, that characterizes the physical system.

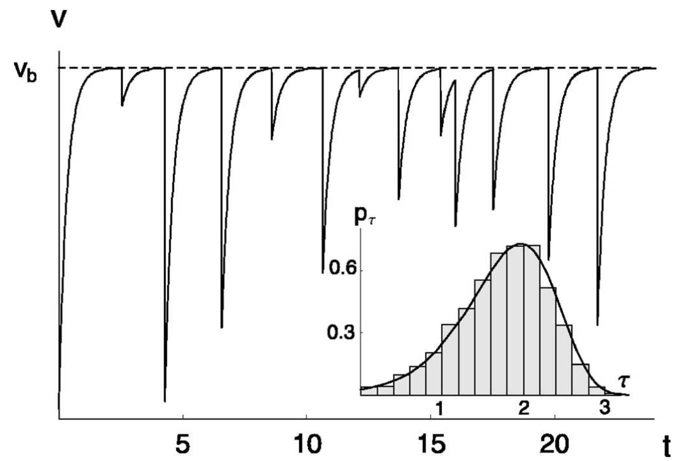

FIG. 2. Example of voltage series for the abstract neuron model with its correspondent pdf of the interarrival time of excitation occurrences (inset). The theoretical $p_\tau(\tau)$ [Eq. (16)] is compared to the histogram numerically obtained using the parameters $k=3$, $k_1=0.003$, $\gamma=1$, and $v_b=2$.

IV. HYPOTHETICAL NEURON MODEL

As an application leading to analytical expressions for the intertime pdf when the jumps bring the system to random values, we analyze a simplified model for the dynamics of the voltage across a nerve membrane. The voltage, v, is assumed to increase exponentially between successive neuron excitations that cause instantaneous drops (y_i) in the voltage level. Such excitations occur randomly, becoming more probable as v approaches a threshold v_b (see Fig. 2). Therefore, we assume that the rate of excitation occurrence depends on v according to an age-specific failure rate of the form $\lambda(v)=k_1/(v_b-v)$. Such a dependence might be explained by the fact that changes in the neuron potential also alter the properties of the synapses [35], thus influencing the excitation rates (simplified random threshold models of this type were previously adopted in [6,5]). To assure analytical tractability, the voltage reductions, y_i, due to excitations are assumed to be exponentially distributed and state independent, so that the jump distribution reduces to $h(y)=\gamma\exp(-\gamma y)$ [e.g., $h(z;x-z)$ in Eq. (2) is simply $h(z)$]. The equation driving the voltage dynamics is thus

$$\frac{dv}{dt} = k(v_b - v) - F(v,t), \tag{10}$$

and the corresponding differential Chapman-Kolmogorov forward equation is

$$\frac{\partial}{\partial t}p(v,t) = -k\frac{\partial}{\partial v}((v_b-v)p(v,t)) - \frac{k_1}{v_b-v}p(v,t) + \int_{-\infty}^{+\infty}\frac{k_1}{v_b-(v-z)}p(v-z,t)\gamma\exp(-\gamma z)dz. \tag{11}$$

Following [1,7], the steady-state probability distribution of v, $p(v)$, can be obtained as follows:

$$p(v) = \frac{C}{v_b - v} \exp\left[-\gamma(v_b - v) - \frac{k_1}{k(v_b - v)} \right]. \quad (12)$$

According to Eq. (3), the distribution of the points before a jump is

$$p_e(v) = \frac{C_e}{(v_b - v)^2} \exp\left[-\gamma(v_b - v) - \frac{k_1}{k(v_b - v)} \right], \quad (13)$$

which, using Eq. (4), gives $p_s(v)$ as follows:

$$p_s(v) = C_s \exp\left[-\gamma(v_b - v) - \frac{k_1}{k(v_b - v)} \right]. \quad (14)$$

Given $p_s(v)$, Eq. (5) can be solved using the method of characteristics [36] to obtain

$$p_\lambda(v,t) = C_\lambda \exp\left[kt - \gamma(v_b - v)e^{kt} - \frac{k_1}{k(v_b - v)} \right]. \quad (15)$$

In the previous equations, C, C_e, C_s, and C_λ are normalization constants for the respective distributions. Inserting Eq. (15) in Eqs. (6) and (7), one finally obtains

$$p_\tau(\tau) = \frac{ke^{k\tau/2}}{K_1(2C_\tau)}[e^{k\tau/2}C_\tau K_0(2e^{k\tau/2}C_\tau) - K_1(2e^{k\tau/2}C_\tau) + e^{k\tau/2}C_\tau K_2(2e^{k\tau/2}C_\tau)], \quad (16)$$

where $C_\tau = \sqrt{(\gamma k_1)/k}$ and $K_n(\cdot)$ is the modified Bessel function [37].

Figure 2 shows an example of a time series along with the pdf of the interarrival time of two successive excitations, $p_\tau(\tau)$. The state-dependent nature of the Poisson noise generates a maximum in the pdf, thus producing a characteristic recurrence interval at which excitations are more likely to occur. Even though the model does not account for the refractory period after an excitation [6], the probability of having two immediately successive excitations is considerably low. Such a behavior depends mainly on the value of k_1. In fact, as k_1 decreases, the probability of having an excitation is reduced and consequently becomes more dependent on the difference $v_b - v$. Since after a jump $v_b - v$ is large, the probability of having consecutive excitations in the short time is reduced, so that $p_\tau(0)$ becomes closer to 0 as k_1 diminishes. An analogous effect is induced by a reduction in γ, with a consequent increase of the average amplitude of jumps. In fact, the higher the jumps are, the longer is the time it takes to v to get close to v_b and therefore to jump again.

V. PERSISTENCE IN RAINFALL EVENTS

A significant fraction of the warm season precipitation in continental mid-latitude regions is either originated from local evaporation recycling or triggered by soil moisture feedbacks on the atmospheric boundary layer. Consequently, in such regions there tends to be a dependence of the rainfall regime on the antecedent soil water conditions. Thus, anomalous dry periods early in the season reduce the probability of summer rainfall occurrence, possibly locking the system in a dry condition. On the other hand, high soil water content in spring may enhance summer rainfall and tend to preserve wet conditions [9,10]. Here we analyze the simplified model introduced in [9], in order to determine the effect that a soil-moisture dependence of rainfall frequency has on the distribution of the interarrival times of rainfall events.

Schematically, the soil water balance at the daily time scale is driven by the stochastic equation [9]

$$nZ_r \frac{dx}{dt} = -L(x) + F(x,t), \quad (17)$$

where x is the relative soil moisture content of the portion of soil interested by the root, nZ_r, with n soil porosity and Z_r soil depth, $L(x)$ represents the losses due to evapotranspiration and deep infiltration during interstorm periods, and $F(x,t)$ are the increments in soil moisture due to infiltration by rainfall events, that at the daily time scale can be considered as instantaneous pulses. In order to include the soil-moisture feedback on precipitation, the frequency of rainfall events may be assumed to partly depend on x [9]. Since soil moisture is bounded between 0 (dry soil) and 1 (complete saturation), $f(0)=0$ and the increments of x due to rainfall events are limited at $x=1$ by means of a Dirac delta function at $1-x$ in the jump distribution [7–9]. Accordingly, assuming an exponential distribution for the depths of rainfall events, the pdf of the soil moisture jumps, y_i, due to rainfall becomes

$$h(y;x) = \gamma e^{-\gamma y} + \delta(y - 1 + x)\int_{1-x}^{\infty} \gamma e^{-\gamma u} du, \quad (18)$$

for $0 < y \leq 1-x$ and with $1/\gamma = \alpha/(nZ_r)$, where α is the average amount of water carried by each rainfall event. Equation (2), thus, can be written

$$\frac{\partial}{\partial t}p(x,t) = -\frac{\partial}{\partial x}(f(x)p(x,t)) - \lambda(x)p(x,t) + \int_{-\infty}^{+\infty} \lambda(x-z)p(x - z,t)h(z;x-z)dz, \quad (19)$$

where $f(x) = -L(x)/(nZ_r)$. Following the same lines of the solution of the neural model (Sec. IV), a formal solution of the interarrival time of rainfall events can be obtained for a process with linear decay, $f(x) = -k = \text{const}$ for $x > 0$ and $f(0)=0$, and with rainfall frequency linearly dependent on x, $\lambda(x) = a + bx$ [9].

The steady-state pdf of x is a mixed one with an atom of probability in zero [9],

$$p(x) = C\left(\frac{1}{k} e^{-(\gamma - a/k)x + bx^2/(2k)} + \frac{\delta(x)}{a} \right), \quad (20)$$

where C is a normalization constant. Given their complexity, the expressions of $p_e(x)$, $p_s(x)$, and $p_\lambda(x,t)$ are not reported here. The rainfall intertime pdf, obtained by numerical integration of $-\int_0^1 (dp_\lambda(x,\tau)/d\tau)dx$, according to Eqs. (6) and (7), is reported in Fig. 3.

Figure 3 also shows the comparison between the $p_\tau(\tau)$ of a process with linear state-dependent rainfall frequency, $\lambda = a + bx$, and the $p_\tau(\tau)$ of a process with constant rainfall

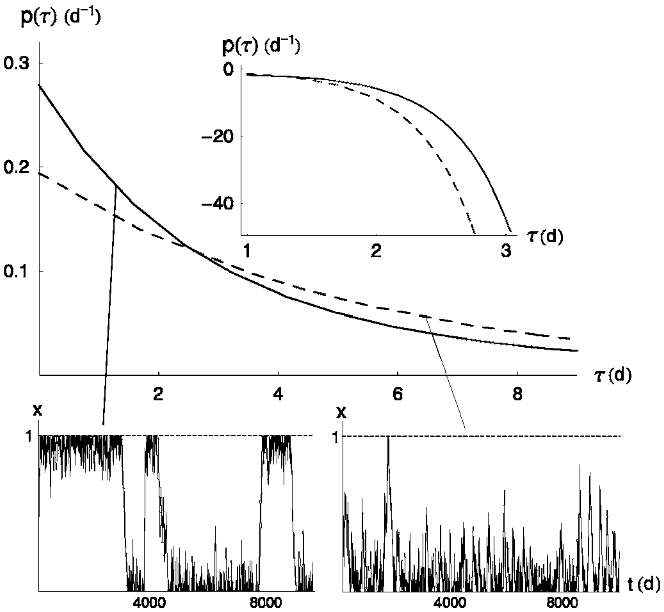

FIG. 3. Examples of $p_\tau(\tau)$ for different hydrological models with constant soil water losses, $f(x)=k$. The different lines refer to a model with state-dependent $\lambda(x)=a+bx$ (continuous line) and constant rainfall frequency $\lambda=\langle\lambda(x)\rangle=a+b\langle x\rangle$ (dashed line). The inset shows a log-log plot of the tails of the two pdf's. The time series at the bottom are two realizations referring to the two different cases. Parameters are $a=0.1d^{-1}$, $b=0.3d^{-1}$, $k=0.0125d^{-1}$, $\langle\lambda(x)\rangle=0.194d^{-1}$, and $\gamma=20$.

frequency equal to the average of the state-dependent process, $\langle\lambda(x)\rangle=a+b\langle x\rangle$. As shown by the time series, the state-dependent Poisson noise is responsible for persistence of dry and wet states which manifests itself in the bi-modality of the steady-state pdf of x. Such a bi-modality is closely related to the timing of rainfall occurrences. In fact, when x is large, the probability of having a rainfall event increases so that the system tends to remain close to $x=1$ (saturation). Therefore, the probability of having frequent rainfall events in a short time is higher than that of a similar process with constant rainfall frequency equal to $\langle\lambda(x)\rangle$. On the other hand, when x decreases, rainfall events become less frequent so that the system tends to maintain low soil moisture values for longer periods. This situation of persistent drought periods explains the slower decay of the tail of $p_\tau(\tau)$ for the state-dependent rainfall process.

VI. CONCLUSIONS

We introduced a general method to determine the pdf of the times between successive jumps of steady-state systems driven by state-dependent Poisson processes that cause instantaneous jumps with random timing and amplitude.

The dynamics of such systems present interesting properties that are discussed through different examples. A possible model for human action execution presented in Sec. III shows the capability of the state-dependent Poisson process to generate power-law tails in the jump intertime pdf. The state dependence of the Poisson noise may also be the cause of time recurrence of jumps, as suggested by the maximum in $p_\tau(\tau)$ obtained in the neuron model (Sec. IV). Finally, the state-dependence of the Poisson process can also explain possible persistent behaviors in the dynamics of jump events and the presence of preferential states in the dynamics of x (Sec. V). The flexibility and the variety of behaviors shown by the state-dependent Poisson process make it a good candidate for modeling many physical phenomena characterized by jump transition.

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